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Research Paper

# Simulation and Analysis of Black Hole Thermodynamics in Modified Gravity Models

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## Abstract

Black hole thermodynamics has emerged as a pivotal field in understanding the fundamental laws governing gravitation and quantum mechanics. Modified gravity models, proposed as extensions or alternatives to General Relativity, offer promising frameworks to resolve anomalies observed in cosmology and astrophysics. This paper presents a comprehensive simulation and analysis of black hole thermodynamics within various modified gravity frameworks, including f(R) gravity, Gauss-Bonnet gravity, and Horava-Lifshitz gravity. Utilizing numerical methods and theoretical modeling, we investigate the temperature, entropy, heat capacity, and phase transitions of black holes under these modified theories. Our results demonstrate distinct thermodynamic behavior compared to classical General Relativity, highlighting modifications in stability conditions and critical phenomena. This study provides insights into the viability of modified gravity theories and their implications for black hole physics, potentially bridging gaps between gravitational theory and quantum field effects.

**Keywords**: Black hole thermodynamics, Modified gravity, f(R) gravity, Gauss-Bonnet gravity, Horava-Lifshitz gravity, Phase transitions, Entropy, Numerical simulation, Stability analysis, Quantum gravity.

## Introduction

Black holes have long been regarded as one of the most fascinating and enigmatic phenomena predicted by Einstein's theory of General Relativity (GR). Their study not only tests the limits of classical gravity but also provides crucial insights into the interface between gravitation, quantum mechanics, and thermodynamics. The discovery that black holes possess thermodynamic properties—most notably temperature and entropy ushered in a new paradigm in theoretical physics, suggesting profound connections between gravity, thermodynamics, and information theory (Bekenstein, 1973; Hawking, 1975).

Despite the tremendous success of General Relativity in explaining gravitational phenomena at large scales, numerous cosmological observations—such as dark energy, dark matter, and the accelerated expansion of the universe—have motivated the development of modified gravity theories. These models extend or modify the Einstein-Hilbert action in an effort to address the shortcomings of GR without invoking exotic matter components (Clifton et al., 2012; Nojiri & Odintsov, 2017). Modified gravity frameworks, such as f(R) gravity, Gauss-Bonnet gravity, and Horava-Lifshitz gravity, have been extensively studied for their theoretical richness and phenomenological implications.

In the context of black hole physics, modified gravity theories often predict alterations in the structure, dynamics, and thermodynamics of black holes (Cognola et al., 2005; Cai et al., 2010). The thermodynamic behavior of black holes—such as temperature, entropy, and heat capacity—may deviate significantly from predictions of classical GR, providing potential observational signatures and constraints on these alternative theories (Myers & Simon, 1988; Hendi et al., 2015). Moreover, the study of phase transitions and stability in black holes under modified gravity is crucial for understanding the underlying microscopic degrees of freedom and their quantum gravitational nature (Kubiznak & Mann, 2012).

This paper aims to simulate and analyze the thermodynamic properties of black holes within several prominent modified gravity models. Using numerical simulations and analytical techniques, we explore how modifications to the gravitational action influence black hole temperature, entropy, heat capacity, and critical phenomena. By systematically comparing results across different theories, we seek to elucidate the distinctive thermodynamic characteristics imparted by modified gravity and discuss their implications for both theoretical physics and potential astrophysical observations.

## Literature Review

The study of black hole thermodynamics has its roots in the pioneering works of Bekenstein (1973) and Hawking (1975), who proposed that black holes possess entropy proportional to the area of their event horizon and emit thermal radiation, respectively. These groundbreaking concepts laid the foundation for an intricate relationship between gravity, quantum mechanics, and thermodynamics, which has continued to evolve over the decades (Bardeen et al., 1973). The classical laws of black hole mechanics, formulated by

Bardeen, Carter, and Hawking, analogize the four laws of thermodynamics, thus positioning black holes as thermodynamic objects with well-defined temperature, entropy, and energy (Wald, 1994).

While General Relativity successfully describes many aspects of black holes, several unresolved issues motivate exploration beyond Einstein's theory. The challenges include explaining cosmic acceleration, dark matter phenomena, and singularity problems, which classical GR cannot adequately resolve (Clifton et al., 2012). Modified gravity theories offer a broader landscape of gravitational dynamics by extending the Einstein-Hilbert action. Among these, f(R) gravity modifies the Ricci scalar term, allowing for richer curvature-dependent dynamics (De Felice & Tsujikawa, 2010). Gauss-Bonnet gravity introduces higher-order curvature invariants, affecting black hole solutions especially in higher dimensions (Nojiri & Odintsov, 2011). Horava-Lifshitz gravity, inspired by condensed matter physics, breaks Lorentz invariance at high energies, providing a potentially renormalizable quantum gravity model (Horava, 2009).

Thermodynamics of black holes in modified gravity frameworks has been extensively studied. For instance, Cognola et al. (2005) investigated black hole entropy and temperature in f(R) models, showing deviations from the area law due to curvature corrections. Cai et al. (2010) examined black holes in Gauss-Bonnet gravity, identifying modifications to thermodynamic stability and phase transitions that depend on the coupling constants of higher-order terms. Similarly, studies in Horava-Lifshitz gravity revealed novel horizon structures and thermodynamic behaviors, highlighting differences from GR black holes (Kiritsis & Kofinas, 2009).

Moreover, the concept of black hole phase transitions, analogous to classical thermodynamic systems, has garnered significant interest. Kubiznak and Mann (2012) pioneered the extended phase space approach, treating cosmological constant as thermodynamic pressure, leading to insights about critical behavior and Van der Waals-like transitions in AdS black holes. Such analyses have been adapted to modified gravity contexts, revealing intricate phase diagrams influenced by gravitational modifications (Hendi et al., 2015; Altamirano et al., 2013).

Numerical simulations and analytical modeling have become indispensable tools for exploring these complex thermodynamic features. Researchers have applied methods like the Euclidean action approach, path integral techniques, and perturbative expansions to derive black hole thermodynamic quantities in modified theories (Jacobson & Myers, 1993; Banerjee & Majhi, 2011). These studies emphasize the sensitivity of black hole thermodynamics to underlying gravitational dynamics and the importance of precise numerical analysis for validating theoretical predictions.

In summary, the existing literature underscores that black hole thermodynamics serves as a fertile ground to test and constrain modified gravity models. The deviations in temperature, entropy, and phase behavior provide both theoretical challenges and opportunities to deepen our understanding of gravity's quantum aspects. However, comprehensive simulations comparing multiple modified gravity frameworks remain sparse, motivating the current study to fill this gap through systematic numerical and analytical investigations.

## Methodology

This study aims to simulate and analyze the thermodynamic properties of black holes within selected modified gravity models—namely f(R) gravity, Gauss-Bonnet gravity, and Horava-Lifshitz gravity. The methodology involves both analytical derivations and numerical simulations to investigate key thermodynamic quantities such as temperature, entropy, heat capacity, and phase transitions. The following subsections outline the theoretical framework, computational approach, and parameters used.

### Theoretical Framework

#### *Black Hole Solutions in Modified Gravity*

Each modified gravity theory considered in this study introduces corrections to the classical Einstein-Hilbert action, which in turn modify black hole solutions and their thermodynamics.

- **f(R) Gravity:** The action is generalized from the Ricci scalar RRR to an arbitrary function f(R)f(R)f(R), leading to field equations that include higher-order derivatives of the metric tensor. The black hole metric solutions used here follow the spherically symmetric, static ansatz derived from these modified field equations (Sotiriou & Faraoni, 2010).
- **Gauss-Bonnet Gravity:** This includes a Gauss-Bonnet term G=R2−4RμνRμν+RμνρσRμνρσ\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}G=R2−4RμνRμν+RμνρσRμνρσ, which affects the gravitational dynamics in higher dimensions (usually five or more). We focus on black hole solutions incorporating the Gauss-Bonnet coupling parameter α\alphaα, which controls the strength of higher curvature effects (Cai, 2002).
- **Horava-Lifshitz Gravity:** Characterized by anisotropic scaling between time and space, this theory modifies the ultraviolet behavior of gravity, resulting in new static black hole solutions with altered horizon structures (Lu, Mei, & Pope, 2009).

***Thermodynamic Quantities***

The primary thermodynamic quantities analyzed are:

- **Hawking Temperature (THT_HTH)**: Computed from the surface gravity κ\kappaκ at the event horizon rhr_hrh:
  TH=κ2π=14πdgttdr|r=rhT_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \left. \frac{d g_{tt}}{dr} \right|_{r=r_h}TH=2πκ=4π1drdgttr=rh

where gttg_{tt}gtt is the time-time component of the metric tensor.

- **Entropy (SSS)**: In modified gravity, entropy does not always follow the Bekenstein-Hawking area law. Wald's entropy formula, which generalizes entropy for higher curvature terms, is employed (Wald, 1993):

S=−2π∫H∂L∂Rμνρσϵμνϵρσh dD−2xS = -2\pi \int_{\mathcal{H}} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \sqrt{h} \, d^{D-2}xS=−2π∫H∂Rμνρσ∂LϵμνϵρσhdD−2x

where L\mathcal{L}L is the Lagrangian density, H\mathcal{H}H is the horizon, and ϵμν\epsilon_{\mu\nu}ϵμν are binormal vectors.

- **Heat Capacity (CCC)**: Defined as

C=dMdTHC = \frac{dM}{dT_H}C=dTHdM

where MMM is the black hole mass. The sign of CCC indicates thermodynamic stability; positive CCC corresponds to stability.

- **Phase Transitions**: Critical points are identified where heat capacity diverges, signaling phase transitions analogous to liquid-gas transitions in classical thermodynamics.

### Numerical Simulation Approach

Due to the complexity of modified gravity field equations, closed-form analytical expressions for thermodynamic variables are not always attainable. Numerical methods are employed for:

- **Locating Event Horizons:** The root-finding algorithms (e.g., Newton-Raphson) are used to solve metric functions for horizon radius rhr_hrh.
- **Calculating Temperature and Entropy:** Numerical differentiation and integration techniques are applied to evaluate surface gravity and Wald entropy integrals.
- **Heat Capacity and Stability Analysis:** Numerical derivatives of mass versus temperature curves are computed to determine heat capacity and identify critical points.

All computations are performed using Python with libraries such as NumPy and SciPy. Symbolic manipulation and verification are done using Mathematica to cross-check analytical steps.

### Parameter Selection

Parameters are chosen based on ranges common in literature and physical plausibility:

- For f(R) gravity, polynomial forms f(R)=R+αR2f(R) = R + \alpha R^2f(R)=R+αR2 are considered with α\alphaα varying between 000 and 10−210^{-2}10−2 in natural units.
- Gauss-Bonnet coupling α\alphaα is varied from 000 (recovering GR) up to 0.10.10.1, consistent with stability requirements.
- Horava-Lifshitz scaling parameters are selected following conventions in existing studies (Lu et al., 2009), focusing on the detailed balance condition.

### Validation and Benchmarking

To validate our simulation framework, results are first benchmarked against known thermodynamic properties of Schwarzschild and Reissner-Nordström black holes within classical General Relativity. Subsequent results from modified gravity are compared to published findings to ensure consistency.

## Results

This section presents the outcomes of the simulations and analyses of black hole thermodynamics within the frameworks of f(R) gravity, Gauss-Bonnet gravity, and Horava-Lifshitz gravity. Key thermodynamic quantities such as temperature, entropy, heat capacity, and phase transitions are systematically explored and compared to classical General Relativity (GR) black holes.

### Black Hole Temperature

The Hawking temperature THT_HTH was computed numerically for a range of horizon radii rhr_hrh and modified gravity parameters. For the baseline Schwarzschild black hole in GR, THT_HTH varies inversely with horizon radius as expected:

TH=14πrh.T_H = \frac{1}{4 \pi r_h}.TH=4πrh1.

- **f(R) Gravity:** Introducing the quadratic term αR2\alpha R^2αR2 leads to subtle corrections in temperature. As α\alphaα increases, the temperature profile deviates slightly, especially at smaller horizon radii, indicating higher temperatures than predicted by GR for small black holes . This suggests modified evaporation rates under quantum corrections (Sotiriou & Faraoni, 2010).
- **Gauss-Bonnet Gravity:** The presence of the coupling parameter α\alphaα causes notable deviations in temperature. For positive α\alphaα, black holes exhibit a non-monotonic temperature behavior with local maxima and minima, indicative of possible thermodynamic phase transitions (Cai, 2002). The critical radius where temperature peaks shift depending on the value of α\alphaα.
- **Horava-Lifshitz Gravity:** Black holes in this framework show distinct temperature curves, with temperature increasing sharply near smaller horizons and plateauing for larger rhr_hrh. The anisotropic scaling alters horizon thermodynamics significantly, confirming predictions by Lu et al. (2009).

### Entropy Behavior

Entropy calculations using Wald's formula reveal that black hole entropy generally deviates from the classical area law S=A4S = \frac{A}{4}S=4A.

- In **f(R) gravity**, entropy includes corrections proportional to derivatives of f(R)f(R)f(R). Numerical integration shows entropy increases faster than area for larger α\alphaα, suggesting enhanced microscopic degrees of freedom (Cognola et al., 2005).
- **Gauss-Bonnet gravity** results indicate entropy contains an additive term proportional to the Gauss-Bonnet invariant, increasing total entropy beyond the classical prediction (Myers & Simon, 1988). This is particularly pronounced for higher-dimensional extensions.
- For **Horava-Lifshitz gravity**, entropy expressions become more complex due to modified horizon geometry. Numerical results suggest a non-linear relation between entropy and horizon area, consistent with previous studies (Kiritsis & Kofinas, 2009).

### Heat Capacity and Stability

Heat capacity CCC curves reveal stability characteristics of black holes:

- **Schwarzschild black holes** exhibit negative heat capacity, confirming classical thermodynamic instability.
- In **f(R) gravity**, increasing α\alphaα softens the negative heat capacity regime, introducing stable regions for intermediate horizon sizes. This suggests modified gravity can stabilize small black holes thermodynamically (Figure 4).
- **Gauss-Bonnet black holes** demonstrate rich phase structure with multiple sign changes in heat capacity, corresponding to first-order phase transitions between small and large black hole phases (Kubiznak & Mann, 2012). These transitions occur near critical coupling values of α\alphaα (Figure 5).
- **Horava-Lifshitz gravity** black holes show predominantly positive heat capacity for a wide range of parameters, implying enhanced thermodynamic stability due to high-energy corrections.

### Phase Transitions

Critical phenomena were identified through divergence points in heat capacity:

- The **Van der Waals-like phase transitions** appear prominently in Gauss-Bonnet gravity, consistent with extended phase space thermodynamics where cosmological constant acts as pressure (Altamirano et al., 2013).
- **f(R) gravity** phase transitions are more subtle, with the phase diagram showing crossover behaviors rather than sharp transitions.
- **Horava-Lifshitz gravity** features modified critical points shifted by anisotropic scaling parameters.

**Table 1.** *Comparative Summary*

| Model | Temperature Behavior | Entropy Deviation | Heat Capacity Stability | Phase Transition Features |
|---|---|---|---|---|
| General Relativity | TH∼1/rhT_H \sim 1/r_hTH∼1/rh | S=A/4S = A/4S=A/4 | Negative, unstable | None |
| f(R) Gravity | Slight increase at small rhr_hrh | Enhanced over area law | Stable regions appear | Subtle crossovers |
| Gauss-Bonnet | Non-monotonic with critical points | Additive curvature term | Multiple stable/unstable zones | First-order Van der Waals-like |
| Horava-Lifshitz | Plateau at large rhr_hrh | Non-linear area relation | Predominantly stable | Shifted critical points |

*Note. Table 1* Summarizes key thermodynamic quantities across the gravity models for a representative set of parameters.

## Discussion

The results presented reveal significant insights into how modified gravity theories impact black hole thermodynamics, offering richer structures compared to classical General Relativity (GR). This discussion interprets the findings in light of theoretical expectations, physical implications, and connections to ongoing research.

### Implications of Temperature Modifications

The observed deviations in Hawking temperature across modified gravity models highlight how corrections to the Einstein-Hilbert action alter black hole evaporation dynamics. In f(R) gravity, the slight increase in temperature at smaller horizon radii suggests that quantum gravitational effects encoded in higher-order curvature terms accelerate black hole evaporation in early stages. This aligns with prior works showing that R2R^2R2 corrections act as effective energy sources near the horizon (Sotiriou & Faraoni, 2010).

Gauss-Bonnet gravity introduces even more dramatic changes. The non-monotonic temperature profile with local extrema indicates black holes can undergo thermal phases where they heat up or cool down as their size changes, reminiscent of liquid-gas systems. Such behavior supports the analogy between black hole thermodynamics and classical thermodynamic systems, reinforcing the interpretation of the Gauss-Bonnet coupling as a parameter controlling microscopic interactions beyond GR (Cai, 2002; Kubiznak & Mann, 2012).

In Horava-Lifshitz gravity, the temperature plateau for large horizons reflects the anisotropic scaling's influence on infrared gravitational dynamics. The modified scaling relations suggest that black hole thermodynamics at low energies differs significantly from classical predictions, which could have implications for understanding quantum gravity phenomenology at astrophysical scales (Lu et al., 2009).

### Entropy Corrections and Microscopic Degrees of Freedom

The departure from the classical area law across all modified models emphasizes the role of additional geometric and topological terms in the gravitational action in encoding microscopic degrees of freedom. Wald's entropy formalism, which accounts for higher curvature contributions, confirms that black hole entropy is sensitive to the underlying gravitational theory.

In f(R) gravity, the enhanced entropy growth with coupling strength α\alphaα may reflect increased quantum fluctuations or new field degrees of freedom interacting near the horizon (Cognola et al., 2005). Similarly, the additive Gauss-Bonnet term suggests topological effects contribute to the entropy budget, potentially linked to string-theoretic corrections (Myers & Simon, 1988).

Horava-Lifshitz gravity's non-linear entropy-area relation hints at a more complex microstructure, possibly due to the breaking of Lorentz invariance at high energies. These corrections could influence black hole information paradox resolutions and entropy bounds in quantum gravity scenarios (Kiritsis & Kofinas, 2009).

### Stability and Phase Transition Phenomena

Heat capacity analysis reveals that modified gravity can stabilize black holes thermodynamically. Classical Schwarzschild black holes are thermodynamically unstable due to negative heat capacity, leading to runaway evaporation. The emergence of positive heat capacity regions in f(R) and Horava-Lifshitz gravity models opens the possibility of stable black hole remnants or equilibrium states.

Gauss-Bonnet black holes' rich phase structure, including first-order phase transitions resembling Van der Waals fluids, corroborates a growing body of work interpreting black holes as thermodynamic systems with rich phase diagrams (Altamirano et al., 2013). These transitions may manifest physically in gravitational wave signatures or black hole mergers, suggesting observational tests of modified gravity.

The presence of critical points shifted by model parameters illustrates how coupling constants control black hole microphysics. This tunability could have cosmological implications, as primordial black holes formed under modified gravity may exhibit distinct thermodynamic lifetimes and evaporation behaviors.

### Limitations and Model Assumptions

While the results are promising, several limitations must be acknowledged:

- The study considers static, spherically symmetric black holes, excluding rotation and charge which can qualitatively alter thermodynamics.
- Numerical methods impose discretization errors and parameter range constraints; extremely large couplings may lead to unphysical solutions or instabilities.
- Theoretical assumptions such as the detailed balance condition in Horava-Lifshitz gravity may restrict generality.

Future work should extend analyses to rotating black holes, incorporate quantum corrections explicitly, and explore observational consequences.

## Conclusion

This study has systematically simulated and analyzed black hole thermodynamics within several prominent modified gravity frameworks: f(R) gravity, Gauss-Bonnet gravity, and Horava-Lifshitz gravity. The investigation revealed that modifications to the Einstein-Hilbert action introduce significant changes in key thermodynamic quantities such as temperature, entropy, heat capacity, and phase transition behavior compared to classical General Relativity (GR).

The Hawking temperature exhibited model-dependent deviations, with f(R) gravity producing modest increases at small horizon scales, Gauss-Bonnet gravity leading to complex non-monotonic temperature profiles indicating rich phase structures, and Horava-Lifshitz gravity yielding plateaus due to anisotropic scaling effects. Entropy calculations showed consistent departures from the classical area law, reflecting additional geometric and topological contributions inherent to each modified gravity theory. Notably, heat capacity analyses identified stable thermodynamic regimes in f(R) and Horava-Lifshitz models, while Gauss-Bonnet black holes displayed Van der Waals-like phase transitions with well-defined critical points.

These findings underscore the profound impact of higher-order curvature corrections and anisotropic scaling on black hole microphysics, thermodynamic stability, and phase behavior. They also reinforce the analogy between black hole thermodynamics and classical thermodynamic systems, suggesting that modified gravity parameters can act as tunable controls for black hole phase transitions and stability properties.

Despite limitations related to assumptions of static, spherically symmetric black holes and numerical constraints, the study provides a comprehensive foundation for future research. The thermodynamic characteristics revealed here have potential implications for quantum gravity, black hole evaporation, and observational signatures in astrophysical black holes.

In summary, the simulation and analysis demonstrate that black hole thermodynamics in modified gravity models offer a fertile ground for exploring new physics beyond General Relativity, with important theoretical and phenomenological consequences.

## Future Research

Building on the insights gained from this study, several avenues for future research emerge to deepen understanding and broaden the scope of black hole thermodynamics in modified gravity models:

### Extension to Rotating and Charged Black Holes

This work focused primarily on static, spherically symmetric black holes. Future studies should investigate rotating (Kerr-like) and charged (Reissner-Nordström-like) black holes within modified gravity frameworks. Rotation and charge introduce additional parameters and can significantly alter thermodynamic behavior, stability, and phase structures (Caldarelli et al., 2000). Extending simulations to these more general cases will provide a more complete thermodynamic characterization.

### Quantum Corrections and Backreaction Effects

Incorporating explicit quantum gravitational corrections beyond semi-classical approximations, including backreaction effects of Hawking radiation on black hole metrics, would enhance the physical realism of the models. Approaches such as loop quantum gravity or effective field theory techniques may be employed to study entropy quantization and evaporation dynamics more precisely (Ashtekar & Bojowald, 2005).

### Dynamical and Non-Equilibrium Thermodynamics

Real astrophysical black holes are dynamical systems often far from equilibrium. Investigating non-equilibrium thermodynamics, black hole formation, and merger processes in modified gravity contexts can illuminate transient thermodynamic phenomena and possible observational signatures (Hiscock & Lindblom, 1985). Numerical relativity simulations incorporating modified gravity terms would be beneficial here.

### Higher-Dimensional and Braneworld Scenarios

Extending analysis to higher-dimensional spacetimes and braneworld models, where Gauss-Bonnet and other higher curvature terms naturally arise, can reveal richer thermodynamic phase diagrams and holographic dualities (Emparan & Reall, 2008). This direction connects black hole thermodynamics with string theory and AdS/CFT correspondence.

### Observational Signatures and Astrophysical Tests

Future work should explore potential observational consequences of modified gravity black hole thermodynamics, such as effects on gravitational wave emission, black hole shadow properties, or accretion disk spectra. Comparing these predictions with data from LIGO-Virgo, the Event Horizon Telescope, and future observatories could provide empirical constraints on modified gravity parameters (Cardoso et al., 2019).

### Machine Learning and Data-Driven Approaches

Leveraging machine learning techniques to classify thermodynamic phases and predict critical points from large parameter spaces could accelerate discovery and improve precision in complex modified gravity models. Such data-driven methods complement analytical and numerical techniques in exploring black hole microphysics (Mehta et al., 2022).

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## Appendix

### Mathematical Details of Modified Gravity Models

f(R) Gravity:The action in f(R) gravity generalizes the Einstein-Hilbert action by replacing the Ricci scalar RRR with a function f(R)f(R)f(R):

S=116πG∫d4x−g f(R)+Smatter,S = \frac{1}{16 \pi G} \int d^4x \sqrt{-g} \, f(R) + S_{\text{matter}},S=16πG1∫d4x−gf(R)+Smatter,

where ggg is the determinant of the metric tensor and SmatterS_{\text{matter}}Smatter is the matter action. The field equations follow from variation with respect to the metric:

f′(R)Rμν−12f(R)gμν+(gμν□−∇μ∇ν)f′(R)=8πGTμν.f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f'(R) = 8 \pi G T_{\mu\nu}.f′(R)Rμν−21f(R)gμν+(gμν □−∇μ∇ν)f′(R)=8πGTμν.

### Black Hole Temperature Formulae

Hawking Temperature in Schwarzschild spacetime:

TH=ℏc38πGMkB,T_H = \frac{\hbar c^3}{8 \pi G M k_B},TH=8πGMkBℏc3,

where MMM is the black hole mass.

- **Modified Temperature in f(R) gravity:**
  The temperature depends on the modified metric function f(r)f(r)f(r) and its derivative at the horizon radius rhr_hrh:

T=14πdf(r)dr∣r=rh.T = \frac{1}{4 \pi} \left. \frac{d f(r)}{dr} \right|_{r = r_h}.T=4π1drdf(r)r=rh.

### Entropy Calculations Using Wald's Formula

The Wald entropy formula for a diffeomorphism-invariant theory with Lagrangian LLL is:

S=−2π∫H∂L∂Rμνρσϵμνϵρσh dD−2x,S = -2 \pi \int_{\mathcal{H}} \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \sqrt{h} \, d^{D-2}x,S=−2π∫H∂Rμνρσ ∂LϵμνϵρσhdD−2x, where ϵμν\epsilon_{\mu\nu}ϵμν is the binormal to the horizon surface H\mathcal{H}H, and hhh is the determinant of the induced metric on the horizon.

### Numerical Methods

The simulations employed the Runge-Kutta 4th order method for solving differential equations governing metric functions, with adaptive step sizing to ensure numerical stability and accuracy. Parameter sweeps over coupling constants and horizon radii were performed using parallel computation on a [specify cluster or system].

## Open Access Statement